Combinatorics

Graph Theory

Counting labelled and unlabelled graphs

There are $2^{\binom{n}{2}}$ labelled graphs of order n. The unlabelled graphs of order n correspond to orbits of the action of S_n on the set of labelled graphs. To count then exactly we can use Burnside's Lemma:

$$\#\text{ orbits} = \frac{1}{|G|}\sum_{g\in G} \#\text{ fixed points of } g.$$

Observe that for any unlabelled graph there are at most $n!$ ways to label to graph in order to obtain a labelled graph (though some of these ways may lead to the same labelled graph), so there are at least $2^{\binom{n}{2}}/n!$ labelled graphs. In fact this bound is asymptotically accurate.

Trees, forests and spanning trees

A *forest* is an acyclic graph. A *tree* is a connected, acyclic graph. The following conditions are equivalent:

1. G is a tree.
2. G is minimal connected.
3. G is maximal acyclic.

Hence every connected graph contains a spanning tree.

Every tree T with at least two vertices contains at least two leaves. To prove this consider a maximal path in T. Observe that removing a leaf from a tree results in another tree. Hence by induction we see that a tree of order n has precisely $n-1$ edges, and the following three conditions are equivalent:

1. G is a tree.
2. G is connected and has size $|G|-1$.
3. G is acyclic and has size $|G|-1$.

There are n^{n-2} labelled trees of order n. To prove this, construct a bijection between the set of labelled trees of order n and the set of strings of length $n-2$ on the set $\{1,\ldots,n\}$, as follows:

- *Tree to string.* Repeatedly select the leaf with the smallest label, write down the label of its neighbour and remove the leaf.
- *String to tree.* Repeatedly select the smallest number which does not appear in the string and which has not already been selected, connect the corresponding vertex to the vertex associated to the first element in the string, and remove the first element of the string.

Bipartite graphs and Hall's Theorem

A graph G is bipartite if and only if every cycle in G is of even length. For if G has no cycles of odd length then we can construct a bipartition of G by choosing a vertex v and letting X be the set of vertices of even distance from v and Y be the set of vertices of odd distance from v.

We have the following three forms of Hall's Theorem.

Hall's Theorem. *Let G be a bipartite graph with bipartition X and Y. Then there is a matching from X to Y if and only if $|\Gamma(A)| \geq |A|$ for every subset $A \subseteq X$.*

Corollary 1. *Let G be a bipartite graph with bipartition X and Y, and let $d \in \mathbb{N}$. Then there exists a set of $|X| - d$ independent edges in G if and only if $|\Gamma(A)| \geq |A| - d$ for every subset $A \subseteq X$.*

Corollary 2. *Let G be a bipartite graph with bipartition X and Y, and let $d \in \mathbb{N}$. Then there exists a 'marriage wherein each man gets d wives' if and only if $|\Gamma(A)| \geq d|A|$ for every subset $A \subseteq X$.*

We can reformulate Hall's Theorem in terms of representative of a collection of sets.

Corollary 3. *Let $\mathcal{A} = \{A_1, \ldots, A_m\} \subseteq \mathcal{P}(X)$. Then $\mathcal{A}$ has a set of distinct representatives if and only if*

$$\left| \bigcup_{i \in I} A_i \right| \geq |I|$$

for every subset $I \subseteq [m]$.

Connectivity

The *connectivity* $\kappa(G)$ of a graph G is

$$\kappa(G) = \begin{cases} \min\{|S| \mid S \subseteq V,\ G - S \text{ is disconnected}\} & \text{if } G \text{ is not complete} \\ |G| - 1 & \text{if } G \text{ is complete.} \end{cases}$$

The *local connectivity* $\kappa(a, b; G)$, where $a \neq b \in V(G)$ and $ab \notin E(G)$, is

$$\kappa(a, b; G) = \min\{|S| \mid S \subseteq V \setminus \{a, b\},\ \text{there is no path from } a \text{ to } b \text{ in } G - S\}.$$

Similarly the *edge connectivity* $\lambda(G)$ is

$$\lambda(G) = \min\{|F| \mid F \subseteq E,\ G - F \text{ is disconnected}\},$$

and the *local edge connectivity* $\lambda(a, b; G)$ is

$$\lambda(a, b; G) = \min\{|F| \mid F \subseteq E,\ \text{there is no path from } a \text{ to } b \text{ in } G - F\}.$$

Menger's Theorem relates the local connectivity of a graph to the existence of vertex disjoint path between two vertices.

Menger's Theorem. *Let G be a graph and let $a, b \in V(G)$, where $ab \notin E(G)$. Then there exists a set of $\kappa(a, b; G)$ vertex-disjoint paths from a to b.*

Clearly there cannot exist any more than this number of paths.

There is also an edge form of the theorem.

Corollary. *Let G be a graph and let $a, b \in V(G)$. Then there exists a set of $\lambda(a, b; G)$ edge-disjoint paths from a to b.*

This follows from the vertex form of the theorem by considering an augmented version of the graph whose vertices are the edges in G, and whose edges correspond to incidence in G.

Hamiltonian cycles

A *Hamiltonian cycle* in a graph is a (vertex-disjoint) spanning cycle. A graph is said to be *Hamiltonian* if it contains a Hamiltonian cycle.

Theorem. *Let G be a connected graph of order $n \geq 3$, in which for any two non-adjacent vertices x and y,*

$$d(x) + d(y) \geq k.$$

If $k < n$ then G has a path of length k. If $k = n$ then G has a Hamiltonian cycle.

Proof. Assume G is not Hamiltonian and consider a path $P = v_1 v_2 \cdots v_\ell$ of maximal length. Note that G has no ℓ-cycle. Define

$$S = \{i \mid v_i v_i \in E(G)\}$$
$$T = \{j \mid v_{j-1} v_\ell \in E(G)\}.$$

These must be disjoint subsets of $\{2, \ldots, \ell\}$, so either we have a contradiction, and hence G is Hamiltonian, or else $\ell - 1 \geq k$, and hence we have a path of length k, as required.

Corollary. *If $\delta(G) \geq |G|/2$ then G is Hamiltonian.*

The Turán graph

The *Turán graph* $T_r(n)$ is the unique r-partite graph of order n having maximal size. The size of $T_r(n)$ is denoted $t_r(n)$. Note that

$$t_r(n) \sim \left(1 - \frac{1}{r}\right) \binom{n}{2}.$$

Clearly $T_r(n)$ is K_{r+1}-free. Does there exist a K_{r+1}-free graph with more edges than $T_r(n)$?

Theorem. *Let G be a K_{r+1}-free graph. Then there exists an r-partite graph H with $V(H) = V(G)$ and $d_H(v) \geq d_G(v)$ for all $v \in V$.*

Proof. By induction on r. Let x be a vertex of G of maximal degree, and let $G' = G[\Gamma(x)]$. Then G' is K_r-free, so by induction we can find an $(r-1)$-partite graph H' on vertex set $\Gamma(x)$ such that $d_{H'}(v) \geq d_{G'}(v)$ for all $v \in \Gamma(x)$. Now form H from H' by joining every vertex of $\Gamma(x)$ to every vertex of $G \setminus \Gamma(x)$.

So the answer is no. In fact $T_r(n)$ is the unique maximal K_{r+1}-free graph.

Turán's Theorem. *Let G be a K_{r+1}-free graph of order n and size at least $t_r(n)$. Then $G = T_r(n)$.*

Proof. By induction on n, the cases $n \leq r$ being trivial. Remove edges until exactly $t_r(n)$ remain. Now let x be a vertex of minimal degree in G. Since $\delta(G) \leq \delta(T_r(n))$ we have

$$e(G - x) = t_r(n) - \delta(G) \geq t_r(n) - \delta(T_r(n)) = t_r(n-1),$$

and so by the induction hypothesis, $G - x = T_r(n-1)$. But then since G is K_{r+1}-free, G itself must be r-partite and so, as $e(G) = t_r(n)$, it must be the case that $G = T_r(n)$. Therefore we can't have removed any vertices to begin with.

Set Systems

Chains and antichains

A *chain* is a family $\mathcal{A} \subseteq \mathcal{P}(X)$ such that if $A, B \in \mathcal{A}$ then either $A \subseteq B$ or $B \subseteq A$. An *antichain* or *Sperner system* is a family $\mathcal{A} \subseteq \mathcal{P}(X)$ such that whenever $A, B \in \mathcal{A}$ and $A \subseteq B$, then $A = B$. Sperner's Lemma gives a bound on the size of an antichain.

Sperner's Lemma. *Let $\mathcal{A} \subseteq \mathcal{P}([n])$ be an antichain. Then*

$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Proof. We shall cover $\mathcal{P}([n])$ with $\binom{n}{\lfloor n/2 \rfloor}$ chains. To do this, we construct injections

$$\begin{array}{lll} (i) & f_r : [n]^{(r)} \longrightarrow [n]^{(r+1)} & \text{for all } r < n/2, \text{ such that } a \subset f(a) \text{ for all } a \\ (ii) & f_r : [n]^{(r)} \longrightarrow [n]^{(r-1)} & \text{for all } r > n/2, \text{ such that } a \supset f(a) \text{ for all } a. \end{array}$$

By symmetry, it's enough just to do (i). We need to verify Hall's condition in the bipartite graph with bipartition $([n]^{(r)}, [n]^{(r+1)})$. This follows by a simple counting argument.

Shadows

If $\mathcal{A} \subseteq [n]^{(r)}$ is an r-regular hypergraph, then the *lower shadow* of $\mathcal{A}$ is

$$\partial^- \mathcal{A} = \partial \mathcal{A} = \{b \in [n]^{r-1} \mid b \subset a \text{ for some } a \in \mathcal{A}\}.$$

The Local LYM Inequality. *Let $\mathcal{A} \subseteq [n]^{(r)}$. Then*

$$\frac{|\partial \mathcal{A}|}{\binom{n}{r-1}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}}$$

with equality if and only if $\mathcal{A} = \emptyset$ or $\mathcal{A} = [n]^{(r)}$.

Proof. As in the proof of Sperner's Lemma, we use a simple counting argument. Equality can only hold in the specified cases, since the bipartite graph spanned by the two level sets is connected.

The LYM Inequality. *Let $\mathcal{A} \subseteq \mathcal{P}([n])$ be an antichain, and let $\mathcal{A}_r = \mathcal{A} \cap [n]^{(r)}$. Then*

$$\sum_{r=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}} \leq 1,$$

with equality if and only if $\mathcal{A} = [n]^{(r)}$ for some r.

Proof. Let $\mathcal{B}_r = \mathcal{A}_r \cup \partial \mathcal{A}_{r+1} \cup \cdots \cup \partial^{n-r} \mathcal{A}_n$. Then $\mathcal{B}_r$ is the disjoint union of $\mathcal{A}_r$ and $\mathcal{B}_{r+1}$. So

$$1 \geq \frac{|\mathcal{B}_0|}{\binom{n}{0}} = \frac{|\partial \mathcal{B}_1|}{\binom{n}{0}} + \frac{|\mathcal{A}_0|}{\binom{n}{0}} \geq \frac{|\mathcal{B}_1|}{\binom{n}{1}} + \frac{|\mathcal{A}_0|}{\binom{n}{0}} = \cdots \geq \sum_{r=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}}.$$

Shadows

The lex and colex orders

We will need the following two orders on $\mathcal{P}([n])$:

- *lex.* $A < B$ if $\min(A \triangle B) > \max A$ or $\min(A \triangle B) \in A$.
- *colex.* $A < B$ if $\max(A \triangle B) \in B$.

Compressions

Let $i, j \in [n]$. Then an i–j *compression* is defined as follows. If $A \in [n]^{(r)}$,

$$C_{ij}(A) = \begin{cases} A \cup \{i\} \setminus \{j\} & \text{if } j \in A,\, i \notin A \\ A & \text{otherwise.} \end{cases}$$

If $\mathcal{A} \subseteq [n]^{(r)}$,

$$C_{ij}(\mathcal{A}) = \{C_{ij}(A) \mid A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \mid C_{ij}(A) \in A\}.$$

So $|C_{ij}(\mathcal{A})| = |\mathcal{A}|$.

Some lemmas (without proof):

$|\partial^- C_{ij}(\mathcal{A})| \leq |\partial^- \mathcal{A}|$.

Definition of i–j compressed and left compressed.

For any given r-regular set system $\mathcal{A}$ there exists a left compressed r-regular set system $\mathcal{B}$ with shadow no greater than that of $\mathcal{A}$.

Want to show that a shadow is minimised on the initial segment of colex. So we would be there if it were the case that any left-compressed family were an initial segment of colex. But this isn't the case, so we have to work a bit harder.

Kruskal–Katona:

Gives a lower bound on the size of a lower shadow, which is achieved if the set system is an initial segment of colex (though this condition is not necessary).

UV-compressions. Proof of Kruskal–Katona using UV-compressions.

Upper shadows. Upper shadows are minimised on an initial segment of lex.

Intersecting families

Definition. An intersecting family is bounded in size by 2^{n-1} — obvious!

Erdős–Ko–Rado Theorem: bounds the size of an r-uniform intersecting set system. Equality holds iff the set system is that subset of $X^{(r)}$ all of which contain one particular element.

Multiple intersections?