Representation Theory

Representations

Let G be a group and V a vector space over a field k. A *representation* of G on V is a group homomorphism $\rho : G \to \mathrm{Aut}(V)$. The *degree* (or *dimension*) of ρ is just $\dim V$.

Equivalent representations

Let $\rho : G \to \mathrm{Aut}(V)$ and $\rho' : G \to \mathrm{Aut}(V')$ be two representations of G. Then a *G-linear map* from ρ to ρ' is a linear map $\phi : V \to V'$ such that

$$\phi \circ (\rho(g)) = (\rho'(g)) \circ \phi$$

for all $g \in G$, or equivalently such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\ V' & \xrightarrow[\rho'(g)]{} & V' \end{array}$$

If additionally ϕ is an isomorphism of vector spaces then we say that ϕ is an *isomorphism* from ρ to ρ', and that ρ is *isomorphic* to ρ'. Notice that $\phi : V \to V'$ is an isomorphism from ρ to ρ' iff ϕ^{-1} is an isomorphism from ρ' to ρ, so isomorphism is an equivalence relation.

Notation

If $g \in G$ and $v \in V$, we often write gv instead of $\rho(g)v$. In this notation, $\phi : V \to V'$ is an isomorphism iff

$$g\phi(v) = \phi(gv)$$

for all $g \in G$ and all $v \in V$.

Subrepresentations

If G acts on V, and W is a subspace of V such that $g(W) \subseteq W$ for all $g \in G$, then we say that W is a *subrepresentation* of V.

A subrepresentation of V is *trivial* if it is 0 or V, or *non-trivial* otherwise.

Irreducible and indecomposible representations

A representation is called *irreducible* if it has no non-trivial subrepresentations.

V is a *direct sum* of W and W', written $V = W \oplus W'$, if W and W' are subrepresentations of V and $V = W \oplus W'$ as vector spaces. Given a representation V, we want to break it up into smaller pieces, that is, write is as

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

where each W_i does not break up into smaller pieces.

We say that a representation is *indecomposible* if it is not a direct sum of smaller representations. If V is indecomposible then it is irreducible, but the converse does not follow in general.

Permutation representations

Let G act on a set X. Then the *permutation representation* of G with respect to this action, $k[X]$, is a $|X|$-dimensional vector space over k with basis $\{e_x \mid x \in X\}$. The action of G on this vector space is defined by

$$g e_x = e_{gx}$$

for all $g \in G$ and all $x \in X$. $k[X]$ is never irreducible, for the 1-dimensional subspace spanned by $\sum_{x \in X} e_x$ is invariant under G.

Faithful representations

If $\rho : G \to \mathrm{Aut}(V)$ is a representation then the *kernel* of ρ is $\ker \rho = \{g \in G \mid \rho(g) = \mathrm{id}\}$.

A representation of G is *faithful* if $\ker \rho = \{1\}$; in this case we say that G acts *faithfully* on V, and G is isomorphic to a subgroup of $\mathrm{Aut}(V)$. Note that $\ker \rho \lhd G$, so if G is simple then every non-trivial representation is faithful.

If G is a finite group then it posesses a faithful finite-dimensional representation. For G acts on itself faithfully by left-multiplication; thus the permuation representation $k[G]$ for this action is faithful.

Complete reducibility

Let G be a finite group and V a representation of G over a field of characteristic zero. Then

1. If $W \subseteq V$ is a G-invariant subspace then there exists a G-invariant complement to W.
2. V is irreducible $\iff$ V is indecomposible.

Proof

1. Let W' be any vector space complement to W. Let $\pi : V \to W$ be the projection of V onto W defined by $\pi(w + w') = w$ for all $w \in W$ and $w' \in W'$, and define

   $$\bar{\pi}(v) = |G|^{-1} \sum_{g \in G} g\, \pi\big(g^{-1} v\big) .$$

   Then

   (a) If $v \in V$ then $\bar{\pi}(v) \in W$, and if $w \in W$ then $\bar{\pi}(w) = w$, so $\bar{\pi}$ is a projection onto W.

   (b) $\mathrm{im}\, \bar{\pi} = W$ and $\bar{\pi}|_W = \mathrm{id}$, so

   $$\ker \bar{\pi} \oplus \mathrm{im}\, \bar{\pi} = V.$$

   (c) For all $v \in V$

   $$h\bar{\pi}(v) = |G|^{-1} \sum_{g \in G} hg\, \pi\big(g^{-1} v\big) = |G|^{-1} \sum_{g \in G} (hg)\, \pi\big((hg)^{-1} hv\big) = \bar{\pi}(hv)$$

   so $\bar{\pi}$ is G-linear.

   (d) If $\bar{\pi}(v) = 0$ then $h\bar{\pi}(v) = \bar{\pi}(hv) = 0$, so $\ker \bar{\pi}$ is G-invariant.

   Hence $\ker \bar{\pi}$ is a G-invariant complement to W.

2. This follows easily from (1). □

Characters

For the whole of this sections, all groups will be finite and all representations will be on finite-dimensional vector spaces over $\mathbb{C}$.

Definition

If $\rho : G \to \mathrm{Aut}(V)$ is a represention, the *character* of ρ is the function

$$\begin{aligned} \chi : G &\longrightarrow \mathbb{C} \\ g &\longmapsto \mathrm{tr}(\rho(g)). \end{aligned}$$

Properties of the character

1. χ does not depend on a choice of basis for V.
2. If ρ and ρ' are isomorphic representations then $\chi_\rho(g) = \chi_{\rho'}(g)$ for all $g \in G$.
3. $\chi(1) = \dim V$.
4. $\chi(g) = \chi(hgh^{-1})$ and so χ is constant on conjugacy classes of G.
5. $\chi(g) = \overline{\chi(g^{-1})}$.
6. $\chi_{\rho \oplus \rho'}(g) = \chi_\rho(g) + \chi_{\rho'}(g)$.

The space of class functions

A *class function* on G is a function $f : G \to \mathbb{C}$ which is constant on conjugacy classes of G. So if V is a representation of G over $\mathbb{C}$ then χ is a class function on G. We write $\mathcal{C}_G$ for the set of all class functions on G. This is a complex vector space, with a basis

$$\begin{aligned} \delta_{\mathcal{O}} : G &\longrightarrow \mathbb{C} \\ g &\longmapsto \begin{cases} 1 & \text{if } g \in \mathcal{O} \\ 0 & \text{if } g \notin \mathcal{O} \end{cases} \end{aligned}$$

where $\mathcal{O}$ ranges over the conjugacy classes of G.

We can define a Hermitian inner product on $\mathcal{C}_G$ by

$$\langle f, f' \rangle = |G|^{-1} \sum_{g \in G} f(g)\overline{f'(g)}.$$

If ρ and ρ' are irreducible representations, then

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \text{if } \rho \text{ is isomorphic to } \rho' \\ 0 & \text{if } \rho \text{ is not isomorphic to } \rho'. \end{cases}$$

Thus the irreducible characters form part of an orthonormal basis for $\mathcal{C}_G$, and so the number of distinct irreducible reresentations is at most the number of conjugacy classes of G. In fact, the irreducible characters form an orthonormal basis for $\mathbb{C}_G$, and hence there are precisely as many distinct irreducible representations as there are conjugacy classes of G.

Consequences of orthogonality

If ρ is an arbitrary representation of G with character χ, and $\chi_1, \ldots, \chi_k$ are the distinct irreducible characters, then by complete reducibility

$$\chi = n_1\chi_1 + \cdots + n_k\chi_k.$$

for some $n_i \in \mathbb{N}$. Therefore $\langle \chi, \chi_i \rangle = n_i$ by orthogonality, and so

$$\chi = \sum n_i \chi_i$$

where $n_i = \langle \chi, \chi_i \rangle$. Thus

1. In any decomposition of ρ into a sum of irreducible representations, each irreducible representation occurs the same number of times.
2. If ρ and ρ' are representations of G with the same character, then $\rho \cong \rho'$.
3. With n_i defined as above,
$$\langle \chi, \chi \rangle = \sum n_i^2$$
and so ρ is irreducible iff $\langle \chi, \chi \rangle = 1$.

The regular representation

Any group G acts on itself by left-multiplication. The permutation representation of this action is called the *regular representation* of G. If χ is the character of the regular representation of G then

$$\chi(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\mathbb{C}[G] = (\dim \rho_1)\rho_1 \oplus \cdots \oplus (\dim \rho_k)\rho_k,$$

and in particular

$$|G| = \sum (\dim \rho_i)^2.$$

Column orthogonality

Fix g and $h \in G$. Then

$$\sum_{\chi \text{ irreducible}} \chi(g)\overline{\chi(h)} = \begin{cases} |C_G(h)| & \text{if } g \text{ is conjugate to } h \\ 0 & \text{if } g \text{ is not conjugate to } h. \end{cases}$$

This is a formal consequence of the orthogonality of characters.

Proof of orthogonality

First we need the following lemma.

Theorem (Schur's Lemma)

Suppose that (ρ, V) and (ρ', V') are irreducible representations of G over $\mathbb{C}$, with characters χ and χ' respectively. Suppose that $\phi : V \to V'$ is a G-linear map. Then

1. Either ϕ is an isomorphism or $\phi = 0$.
2. If $\phi : V \to V$ is an isomorphism then ϕ is multiplication by a scalar $\lambda \in \mathbb{C}$, and so

$$\mathrm{Hom}_G(V, V') = \begin{cases} \mathbb{C} & \text{if } \rho \text{ is isomorphic to } \rho' \\ 0 & \text{if } \rho \text{ is not isomorphic to } \rho'. \end{cases}$$

Proof

1. Observe that $\ker \phi$ is a subrepresentation of V and $\operatorname{im} \phi$ is a subrepresentation of V'. Since V and V' are both irreducible, $\ker \phi = 0$ or V and $\operatorname{im} \phi = 0$ or V'. The result follows.
2. Since $\mathbb{C}$ is algebraically closed, ϕ has an eigenvalue λ and an eigenvector v for λ. Then $\tilde{\phi} = \phi - \lambda I$ is also a G-linear map $V \to V$. But $\tilde{\phi}(v) = 0$ and so $\ker \phi \neq 0$. But then since V is irreducible, $\ker \phi = V$ and so $\phi = \lambda I$. □

Now let (ρ, V) and (ρ', V') be as above and let $\phi : V \to V'$ be *any* linear map. Define

$$\mathrm{Av}\, \phi = |G|^{-1} \sum_{g \in G} g^{-1} \phi g.$$

Then $\mathrm{Av}\, \phi$ is a G-linear map. Furthermore, $\mathrm{tr}(\mathrm{Av}\, \phi) = \mathrm{tr}\, \phi$, so in particular if $\mathrm{tr}\, \phi \neq 0$ then $\mathrm{Av}\, \phi \neq 0$.

Now on to the main part of the proof. Choose bases for V and V' and write $\rho(g)$ and $\rho'(g)$ as matrices with respect to these bases. Then

$$\begin{aligned} \langle \chi, \chi \rangle &= |G|^{-1} \sum_{g \in G} \chi(g) \overline{\chi'(g)} \\ &= |G|^{-1} \sum_{g \in G} \mathrm{tr}(\rho(g)) \overline{\mathrm{tr}(\rho(g))} \\ &= |G|^{-1} \sum_{\substack{g \in G \\ i,j}} \rho(g)_{ii} \rho'(g^{-1})_{jj}. \end{aligned}$$

To be continued...

Observe in passing that a consequence of Schur's Lemma and the orthogonality of characters is that if (ρ, V) and (ρ', V') are two representations of G with characters χ and χ', then

$$\dim \mathrm{Hom}_G(V, V') = \langle \chi, \chi' \rangle.$$

Operations on characters

Motivation

Let $f, f' \in \mathcal{C}_G$. Then the following operations are defined:

- inner product: $\langle f, f' \rangle$
- sum: $(f + f')(g) = f(g) + f'(g)$
- involution: $f^*(g) = f(g^{-1})$
- product: $(ff')(g) = f(g)f'(g)$.

Note that $\chi_{\rho\oplus\rho'} = \chi_\rho + \chi_{\rho'}$, so that the sum of characters has a representation theoretic interpretation — but what about the others? We shall define the *dual* of a representation and the *tensor product* of two representations accordingly.

The dual of a representation

If ρ is a representation of G on V, we can make G act on the dual vector space V^*. Define the *dual representation* ρ^* of ρ by

$$(\rho^*(g)\,\phi)(v) = \phi(\rho(g^{-1})\,v).$$

By considering the matrices of ρ and ρ' with respect to a pair of dual bases, we see that the matrix of $\rho^*(g)$ is the transpose of the matrix of $\rho(g^{-1})$, and so $\chi_{\rho^*}(g) = \chi_\rho(g^{-1}) = \chi_\rho^*(g)$.

Note that in general there is no G-linear isomorphism between V and V^*. In fact, $V \cong V^*$ iff $\chi_V = \overline{\chi_V}$, that is, if $\chi_V(g) \in \mathbb{R}$ for all $g \in G$.

The tenor product

Let V and W be vector spaces over k, with bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ respectively. Then we define the *tensor product* of V and W to be the vector space $V \otimes W$ with basis $v_i \otimes w_j$ for $1 \le i \le m$ and $1 \le j \le n$. So $\dim(V \otimes W) = \dim(V)\dim(W)$.

We define a bilinear map $V \times W \to V \otimes W$ by defining it on a basis as

$$(v_i, w_j) \longmapsto v_i \otimes w_j$$

and extending linearly, so

$$\Big(\sum \lambda_i v_i, \sum \mu_j w_j\Big) \longmapsto \sum \lambda_i \mu_j (v_i \otimes w_j).$$

We denote the image of $(v, w) \in V \times W$ by $v \otimes w$.

An important property of tensor products is that for any vector space U over k, there exists a bijection

$$\big\{\text{bilinear maps } V \times W \to U\big\} \longleftrightarrow \big\{\text{linear maps } V \otimes W \to U\big\},$$

and $V \otimes W$ is the unique vector space with this property. We say that 'the tensor product is universal for bilinear mappings'.

If $f : V \to V$ and $f' : W \to W$ are linear maps, we can define a linear map $(f \otimes f')$ as

$$\begin{aligned} V \otimes W &\longrightarrow V \otimes W \\ v_i \otimes w_j &\longmapsto f(v_i) \otimes f'(w_j). \end{aligned}$$

Then

1. $v \otimes w \mapsto f(v) \otimes f(w)$ for all $v \in V$ and all $w \in W$.
2. $\operatorname{tr}(f \otimes f') = \operatorname{tr}(f) \operatorname{tr}(f')$.

Symmetric and exterior powers

We may define a linear map $\sigma : V \otimes V \to V \otimes V$ by

$$v_i \otimes v_j \longmapsto v_j \otimes v_i,$$

so $v \otimes v' \mapsto v' \otimes v$ for all $v, v' \in V$. Then $\sigma^2 = 1$, and so $(\sigma - 1)(\sigma + 1) = 0$. Hence σ has the two eigenvalues ± 1 on $V \otimes V$. Define

$$\begin{aligned} S^2 V &= \{a \in V \otimes V \mid \sigma a = a\} \\ \textstyle\bigwedge^2 V &= \{a \in V \otimes V \mid \sigma a = -a\} \end{aligned}$$

to be the two eigenspaces of σ. These are called the *second symmetric power* and the *second exterior power* of V respectively. $S^2 V$ has the basis

$$v_i v_j = v_i \otimes v_j + v_j \otimes v_i \qquad (1 \le i \le j \le d)$$

and so $\dim(S^2 V) = d(d+1)/2$, where $d = \dim V$. $\bigwedge^2 V$ has the basis

$$v_i \wedge v_j = v_i \otimes v_j - v_j \otimes v_j \qquad (1 \le i < j \le d)$$

and so $\dim(\bigwedge^2 V) = (d-1)d/2$. Hence

$$V \otimes V = S^2 V \oplus \textstyle\bigwedge^2 V.$$

In general, we write $V^{\otimes n}$ to mean $V \otimes \cdots \otimes V$, and for each $\omega \in S_n$ we define a linear map $\sigma_\omega : V^{\otimes n} \to V^{\otimes n}$ by

$$v_{i_1} \otimes \cdots \otimes v_{i_n} \longmapsto v_{i_{\omega(1)}} \otimes \cdots \otimes v_{i_{\omega(n)}}.$$

Then we define

$$\begin{aligned} S^n V &= \{a \in V^{\otimes n} \mid \sigma_\omega a = a \quad \text{for all } \omega \in S_n\} \\ &= \{a \in V^{\otimes n} \mid \sigma_\omega a = a \quad \text{for all } \omega = (i \;\; (i+1))\} \end{aligned}$$

$$\begin{aligned} \textstyle\bigwedge^n V &= \{a \in V^{\otimes n} \mid \sigma_\omega a = \operatorname{sgn}(\omega)\, a \quad \text{for all } \omega \in S_n\} \\ &= \{a \in V^{\otimes n} \mid \sigma_\omega a = -a \qquad \text{for all } \omega = (i \;\; (i+1))\}, \end{aligned}$$

and

$$\dim(S^n V) = \binom{d+1}{n} \qquad \dim(\textstyle\bigwedge^n V) = \binom{d}{n}.$$

The tensor product of two representations

Let (ρ, V) and (ρ', V') be two representations of G. Then the *tensor product* $\rho \otimes \rho'$ of ρ and ρ' is defined by

$$(\rho \otimes \rho')(g) = \rho(g) \otimes \rho'(g).$$

$\rho \otimes \rho'$ is a representation of G on $V \otimes W$.

Further, $S^n V$ and $\bigwedge^n V$ are subrepresentations of $V^{\otimes n}$. In the case $n = 2$,

$$V \otimes V = S^2 V \oplus \bigwedge^2 V$$

and the characters of $S^2 V$ and $\bigwedge^2 V$ are

$$\begin{aligned} \chi_{S^2 V} &= \frac{1}{2}\left[(\chi(g))^2 + \chi(g^2)\right] \\ \chi_{\bigwedge^2 V} &= \frac{1}{2}\left[(\chi(g))^2 - \chi(g^2)\right]. \end{aligned}$$

Induction and Restriction

Induction

Let G be a finite group and let $H \leq G$. Given a representation V of H we define the *induced representation* as

$$\begin{aligned}\operatorname{Ind}_H^G V &= \operatorname{Hom}_H(\mathbb{C}G, V) \\ &= \{f : G \to V \mid f(hg) = hf(g) \text{ for all } h \in H,\, g \in G\},\end{aligned}$$

the space of H-linear maps from $\mathbb{C}G$ to V. G acts on $\operatorname{Ind}_H^G V$ in the following manner: If $x \in G$ and $f : G \to V \in \operatorname{Hom}_H(\mathbb{C}G, V)$ then

$$(x * f)(g) = f(gx).$$

Properties of the induced representation

Let V be a representation of H and let χ be its character. Then

1. $\dim \operatorname{Ind}_H^G V = |H\backslash G| \cdot \dim V = (|G|/|H|) \cdot \dim V$.
2. If χ' is the character of $\operatorname{Ind}_H^G V$ then

$$\chi'(x) = |H|^{-1} \sum_{\substack{g \in G \\ gxg^{-1} \in H}} \chi(gxg^{-1}) = \sum_{\substack{Hg \in H\backslash G \\ Hgx = Hg}} \chi(gxg^{-1}).$$

Restriction

Let G be a finite group and let $H \leq G$. Given a representation W of G we get a representation $\operatorname{Res}_H^G W$ of H just by resricting the domain of the representation to H.

Fröbenius reciprocity

Let V be a representation of H with character χ and let W be a representation of G with character χ'. Then

$$\langle \chi', \operatorname{Ind}_H^G \chi \rangle_G = \langle \operatorname{Res}_H^G \chi', \chi \rangle_H$$

and

$$\operatorname{Hom}_G(W, \operatorname{Ind}_H^G V) = \operatorname{Hom}_H(\operatorname{Res}_H^G W, V).$$

The Mackey formula

See lecture notes.

Compact Groups

A *topological group* is a group which is also a topological space, and where the group operations are continuous maps with respect to this topology. A *compact group* is a topological group which is compact as a topological space.

A *representation* of a topological group is a *continuous* group homomorphism $\rho : G \to \mathrm{Aut}(V)$ for some finite dimensional vector space V. (In fact every representation of a compact group is also differentiable — but we won't prove this.)

Note that every finite group G is a compact group with the discrete topology, and that with this topology every representation of G in the old sense is continuous and so is a representation of G as a topological group.

Haar measures

A linear function $\int_G : \{\text{continuous functions } G \to \mathbb{C}\} \to \mathbb{C}$ given by

$$f(g) \longmapsto \int_G f(g)\, dg$$

is called a *Haar measure* if

1. $\int_G 1\, dg = 1$ (i.e. the "volume" of G is 1), and
2. $\int_G$ is translation invariant, that is $\int_G f(xg)\, dg = \int_G f(g)\, dg$ for all $x \in G$.

Every compact group has a Haar measure.

Properties of representations of compact groups

Using Harr measure instead of averaging over all the elements of the group, the theorems for finite groups carry over to compact groups. In particular, if G is a compact group then:

1. Every finite dimensional representation of G is a sum of irreducible representations.
2. (Schur's Lemma) If ρ and ρ' are irreducible representations of G, then

$$\mathrm{Hom}_G(\rho, \rho') = \begin{cases} \mathbb{C} & \text{if } \rho \cong \rho' \\ 0 & \text{otherwise.} \end{cases}$$

3. Let $\mathcal{C}_G$ be the set of all class functions on G, where we now require a class function to be *continuous*. If ρ is a representation of G then the character χ is a class function and if we define a Hermitian inner product on $\mathcal{C}_G$ by

$$\langle f, f' \rangle = \int_G f(g)\overline{f'(g)}\, dg$$

then the distinct irreducible characters form an orthonormal basis for the Hilbert space $\mathcal{C}_G$. All the consequences of this theorem are also still valid.

4. If G is abelian then every irreducible representation is one-dimensional.
5. If G and H are compact, and V and W are irreducible representations of G and H respectively, then $V \otimes W$ is an irreducible representation of $G \times H$.

The Groups S^1 and SU_2

Representations of S^1

The group
$$S^1 = \mathrm{U}_1(\mathbb{C}) = \{A \in \mathrm{GL}_1(\mathbb{C}) \mid A\bar{A}^T = I\}$$
is a compact group.

The one-dimensional representations of S^1 are the maps $z \mapsto z^n$ for $n \in \mathbb{Z}$, where we identify S^1 as the unit circle in $\mathbb{C}$, and $\mathrm{GL}_1(\mathbb{C})$ with $\mathbb{C}$ itself. These are all the irreducible representations, and every finite dimensional representation is a direct sum of these.

The group SU_2

The group
$$\mathrm{SU}_2 = \{A \in \mathrm{GL}_2(\mathbb{C}) \mid A\bar{A}^T = I,\ \det A = 1\}$$
is a compact group. In fact
$$\mathrm{SU}_2 = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \mid a\bar{a} + b\bar{b} = 1 \right\},$$
and so SU_2 is isomorphic to a 3-sphere.

Conjugacy classes of SU_2

The centre of SU_2 is
$$Z(\mathrm{SU}_2) = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$
Define
$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C},\ a\bar{a} = 1 \right\} \subset \mathrm{SU}_2,$$
the set of diagonal matrices in SU_2. Then $T \cong S^1$ is called a *maximal torus* in SU_2. Every conjugacy class in SU_2 meets T. In fact if $\mathcal{O}$ is a conjugacy class in SU_2 then
$$\mathcal{O} \cap T = \begin{cases} \mathcal{O} & \text{if } \mathcal{O} \subseteq Z(\mathrm{SU}_2) \\ \{x, x^{-1}\} & \text{if } \mathcal{O} \text{ isn't central,} \end{cases}$$
where g has eigenvalues λ and λ^{-1} and
$$x = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$
Thus there exists a bijection between the set of conjugacy classes in SU_2 and the interval $[-1, 1]$, given by
$$g \longmapsto \tfrac{1}{2}\operatorname{tr}(g) = \tfrac{1}{2}(\lambda + \lambda^{-1}).$$
Now let
$$\mathcal{O}_t = \{g \in \mathrm{SU}_2 \mid \tfrac{1}{2}\operatorname{tr}(g) = t\},$$
where $-1 \leq t \leq 1$. Then $\mathcal{O}_t$ is a conjugacy class in SU_2, and these are all the conjugacy classes. If $t = \pm 1$ then $\mathcal{O}_t = \{\pm I\}$ and if $-1 < t < 1$ then $\mathcal{O}_t \cong S^2$.

Representations of SU_2

(f.d.) Representations of SU_2 are precisely polynomials with integer coefficients which are symmetric in z and z^{-1}. The irreducible reps are the ones $z^n + z^{n-2} + z^{n-4} + \cdots + z^{2-n} + z^{-n}$.

Lie Algebras

$\mathfrak{sl}_n = \{n \times n \text{ matrices } A \text{ over } \mathbb{C} \mid \operatorname{tr} A = 0\}$. It is a vector space over $\mathbb{C}$.

$\mathfrak{sl}_n$ is not generally closed under multiplication, but $AB - BA$ is in it. This is the Lie bracket.

$\mathfrak{sl}_2$ has a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We have the relations

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Definition of a Lie algebra and their representations. Examples.

Explanation of Lie algebras

The Lie algebra corresponsing to a group is 'the tangent space at $1 \in G$' and more background/motivational stuff

Relation between reps of groups and reps of their Lie algebras.

The Lie Algebra $\mathfrak{sl}_2$

Irreducible modules: weight spaces, highest weight vectors

Complete reducibility for f.d. reps of $\mathfrak{sl}_2$